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Adhesion Dynamics on the Line: The Mass Aggregation Process

Toufic Suidan[1]

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This paper studies several problems in adhesion dynamics on the real line. We consider an ensemble of particles with i.i.d. initial velocities. In the gravitational case, we identify a shock time and describe several aspects of the mass aggregation process before, at and after this time. We also study the long time behavior of the ballistic aggregation process.

KEY WORDS: Adhesion dynamics; gravitationally interacting sticky particles.

1. INTRODUCTION

The gravitationally interacting sticky particle model was originally proposed by Zeldovich[1] in order to study the problem of large scale structures in the universe. The problem has been given attention in E *et al.*[2] and Martin and Piasecki[3] [MP]; there is also a useful connection with Burgers turbulence.[4] The formulation of the problem we consider originated in [MP], and involves a simplified one-dimensional version of Zeldovich's model; several properties of the ballistic model, for which gravity is turned, are also discussed.

Consider a system of N point masses (particles) on the real line which attract each other with forces proportional to the product of their respective masses and independent of their inter-particle distance. The corresponding N-body Hamiltonian has the form:

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \gamma \sum_{i<j} m_i m_j \, |x_i - x_j| \tag{1}$$

[1] Department of Mathematics, Princeton University, Princeton, New Jersey; e-mail: tmsuidan @math.princeton.edu

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where γ is the gravitational constant, and m_i, x_i, p_i are the mass, position, and momentum of particle i; the potential is the standard one-dimensional gravitational potential. This Hamiltonian governs the dynamics of the particle system for time intervals separating binary collisions. In our case collisions are perfectly inelastic, i.e., momentum is conserved but energy is not. To be more descriptive, when two particles collide a larger particle of mass equal to the sum of the masses of the initial colliding particles is created; the momentum of this new particle is the sum of the momenta of the initial colliding particles. The subsequent dynamics is governed by a Hamiltonian of the above form with different m_i, x_i, and p_i. The rules which determine the dynamics have been completely specified.

The problem of interest, first studied in the paper by Martin and Piasecki, can be described as follows. Let M be the total mass of the above system of particles. At the initial moment of time, $t=0$, there are N identical particles each having mass $m=M/N$; the particles are equally spaced in the interval $[0, 1] \subset R$ and have separation distance $a=1/N$. We need the following useful notation. Consider the cluster of size n beginning at particle $j+1$ (if $j+n \leqslant N$). The center of mass of this cluster of particles $(j+1,..., j+n)$ at time t (before any collisions of particles in this cluster with particles outside this cluster) is denoted $X^n_{j+1}(t)$ and is equal to

$$X^n_{j+1}(t) = \left(j + \frac{n+1}{2} \right) a + \frac{t}{n} \sum_{s=1}^{n} v_{j+s} + \gamma m(N-n-2j) \frac{t^2}{2} \tag{2}$$

where v_i is the initial velocity of particle i. Bonvin *et al.*[5] [BMPZ] noticed that the necessary and sufficient conditions for the formation of an aggregate of mass mn by time t from the cluster $(j+1,..., j+n)$ subject to the constraint that all particles not in this cluster remain a finite distance from this aggregate until at least time t are:

$$X^r_{j+1}(t) > X^{n-r}_{j+r+1}(t), \qquad r = 1, 2,..., n-1$$

$$X^s_{j-s+1}(t) < X^n_{j+1}(t), \qquad s = 1, 2,..., j$$

$$X^n_{j+1}(t) < X^s_{j+n+1}(t), \qquad s = 1, 2,..., N-(n+j)$$

The last condition is only meaningful if $j < N-n$. The first condition states that for any partition of $(j+1,..., j+n)$ into a left and right piece, the center of masses of these two pieces must cross by time t. The second and third conditions merely state that the cluster $(j+1,..., j+n)$ does not have interaction with particles that do not belong to it.

Denote by $\Theta(x)$ the Heaviside function which takes the value 1 for $x>0$ and 0 otherwise. A straightforward substitution of (2) into the above conditions allows us to rewrite these conditions as

$$\prod_{r=1}^{n-1} \Theta\left(\sum_{l=1}^{r} v_{j+l}-\frac{r}{n}\sum_{l=1}^{n} v_{j+l}+\frac{m\tau r(n-r)}{2}\right)=1 \tag{3}$$

$$\prod_{r=0}^{j-1} \Theta\left(\frac{j-r}{(j-r)+n}\sum_{s=1}^{(j-r)+n} v_{r+s}-\sum_{s=1}^{j-r} v_{r+s}-\frac{mn(j-r)\,\tau}{2}\right)=1 \tag{4}$$

$$\prod_{s=1}^{N-(j+n)} \Theta\left(\frac{n}{n+s}\sum_{i=1}^{n+s} v_{j+i}-\sum_{i=1}^{n} v_{j+i}-\frac{mns\tau}{2}\right)=1 \tag{5}$$

where $\tau=\gamma t-(1/\rho t)$ and $\rho=m/a$.

Let $v_1,..., v_N$ be real valued i.i.d. random variables which have densities symmetric about 0; we will also assume that there exists a $\delta>0$ so that these random variables have $4+\delta$ moments. The questions we address are: (i) what probabilistic statements can be made about the aggregation process for the dynamics with gravity; (ii) what such statements can be made about the dynamics when there is no gravity. [BMPZ] studied (i) for the case of Gaussian initial velocity distribution. They discovered a critical time t^* and showed that there were no macroscopic mass aggregates before t^* and that there is a macroscopic mass after t^*. The methods they use involve explicit calculations with the Gaussian distribution and the Sparre–Andersen theorem. [BMPZ] also conjecture that before the critical time only masses of size $\sqrt{N}$ form.

For (i), we identify the same critical time t^*, before which there is a massive aggregate with probability 0 and after which there is a massive aggregate with probability 1; we refer to t^* as the shock time and we note that t^* depends only on γ and ρ (in particular, $t^*=1/\sqrt{\gamma\rho}$ which coincides with $\tau=0$). We also give an exact formula for the probability of having r masses at t^*; this leads to a very simple formula for the expected number of masses at t^*. The first of these statements is to be understood in the continuum limit where $N\to\infty$ in such a way that $M=mN$ and $\rho=m/a$ remain constant. Physically, it is important to note that t^* is exactly the time at which the system with every initial velocity set to 0 would collapse to form one mass aggregate; the fact that this happens for large N illustrates that the dynamics, although random, is truly dominated by gravitational influences. To understand the limiting process more thoroughly, we find a sequence of times τ_N for which the above probabilities can be controlled, uniformly. We also find partial evidence for the [MP] conjecture mentioned above. For (ii), we calculate the probability of having r masses given N initial masses in the limit as $t\to\infty$; it is observed that the ballistic case $(\gamma=0)$

has very different dynamical properties than the gravitational case. The methods used in this paper are geometrically motivated and allow more freedom in the choice of initial velocity distributions, thus displaying the universality of the dynamics.

2. NO MACROSCOPIC MASSES IN CASE $\tau < 0 < \gamma$

Fix $\gamma > 0$, $\varepsilon > 0$ and $\tau < 0$. The probability that an aggregate of mass greater than εM is created by time τ tends to 0 in the continuum limit; actually, the statement of the following theorem is stronger. Let δ be defined as in the introduction, and pick any $\nu \in (1 - \delta/2(6+\delta), 1]$. Define $A_\delta^N(\tau)$ to be the set of initial velocities for which an aggregate of at least mass $\varepsilon m N^{1/(2-\nu)}$ is created by time τ; N refers to the initial number of particles in the system. We prove

Theorem 1. $P(A_\delta^N(\tau)) \to 0$ as $N \to \infty$.

Before beginning the calculations we would like to note that nonsensical statements such as $n/2$ (where n is not necessarily even) will appear often; this is to avoid pedantic notations for quantities such as "greatest integer less than;" sometime these quantities are important, but in this paper they play no role and will be suppressed. We first estimate a related quantity which is denoted $p_N^n(\tau)$ and set equal to

$$P\left(\prod_{r=1}^{n-1} \Theta(X_1^r(t) - X_{r+1}^{n-r}(t)) = 1\right)$$
$$= P\left(\prod_{r=1}^{n-1} \Theta\left(\sum_{l=1}^{r} v_l - \frac{r}{n} \sum_{l=1}^{n} v_l + \frac{m\tau r(n-r)}{2}\right) = 1\right)$$

which is clearly bounded by

$$P\left(\sum_{l=1}^{n/2} v_l - \frac{1}{2} \sum_{l=1}^{n} v_l + \frac{m\tau n^2}{8} > 0\right)$$

For $n > (\varepsilon N)^{1/(2-\nu)}$ we further bound this quantity by

$$P\left(\sum_{l=1}^{n/2} v_l - \frac{1}{2} \sum_{l=1}^{n} v_l + \frac{\varepsilon M \tau n^\nu}{8} > 0\right) \leqslant 2P\left(\left|\sum_{l=1}^{n} v_l\right| > \left(-\frac{\varepsilon M \tau}{16}\right) n^\nu\right)$$
$$\leqslant \mathscr{C}\left(-\frac{16}{\varepsilon M \tau}\right)^{4+\delta} \frac{1}{n^{(4+\delta)(\nu - 1/2)}}$$
$$\leqslant \mathscr{D}\left(-\frac{16}{\varepsilon M \tau}\right)^{4+\delta} \frac{1}{N^{(4+\delta)(\nu-1/2)/(2-\nu)}}$$

where $\mathscr{C}$ is a constant which depends only on the moments of the random variables v_i and $\mathscr{D}$ is a constant which absorbs some factor of ε. The last bound is a combination of Chebyshev's inequality, the Marcinkiewicz–Zygmund inequality and a short argument; the statement of the Marcinkiewicz–Zygmund inequality and the argument mentioned are given in the appendix; these inequalities are used to show that

$$E\left|\sum_{i=1}^{n} v_i\right|^{4+\delta} = O(n^{2+\delta/2}) \tag{6}$$

The probability of having an aggregate of exactly mass $mn > m(\varepsilon N)^{1/(2-\nu)}$ form by time τ can easily be bounded by

$$\sum_{j=1}^{N-n} p_N^n(\tau) = (N-n)\, p_N^n(\tau)$$

This reflects the fact that the n-cluster could begin at $N-n$ particle positions. Sum over all $n > (\varepsilon N)^{1/(2-\nu)}$:

$$\begin{aligned}\sum_{n > (\varepsilon N)^{1/(2-\nu)}} (N-n)\, p_N^n(\tau) &\leqslant N^2 p_N^{(\varepsilon N)^{1/(2-\nu)}}(\tau)\\ &\leqslant \frac{\mathscr{D}(\varepsilon)}{N^{[(\nu-1/2)(4+\delta)/(2-\nu)]-2}}\left(-\frac{16}{\varepsilon M\tau}\right)^{4+\delta}\end{aligned}$$

This last expression tends to 0 as $N \to \infty$. Thus, the probability of having a mass aggregate which is a finite fraction of the total mass M at time $\tau < 0$ tends to 0 as $N \to \infty$; the statement also says something about the non-existence of smaller mass aggregates. Note that we did not need the fact that $v_1,..., v_N$ have densities.

3. MASS AGGREGATION AT THE CRITICAL TIME $\tau = 0$

Let N be the initial number of particles. Define V_r^N to be the set of initial velocities $(v_1,..., v_N) \in R^N$ so that there are exactly r masses at time $\tau = 0$. Define W_n^N to be the set of initial velocities $(v_1,..., v_N) \in R^N$ which have the property: at time $\tau = 0$ there exists an aggregate of mass exactly Mn/N. We prove

Theorem 2. $P(V_r^N) = (1/r!\, N!)(d^N/dz^N)(\log 1/(1-z))^r|_{z=0}$.

Theorem 3. $P(W_n^N) = 1/n$ for $n > N/2$.

Corollary 1. The expected number of masses at t^* is $\sum_{j=1}^{N} 1/j$.

We first note that the case $\tau = 0$ only has meaning when $\gamma > 0$. It will turn out that the key to understanding the statistics of the aggregation process lies in understanding statistical properties of the convex envelope of the symmetric random walk process generated by the real valued i.i.d. random variables $v_1,..., v_N$. To explain this connection we need to define a few concepts. Let $(v_1,..., v_N) \in R^N$. Define $co(v_1,..., v_N)$ to be the convex hull of the curve in R^2 generated by connecting the points $(0, 0)$, $(1, v_1)$, $(2, v_1 + v_2)$,..., $(N, v_1 + \cdots + v_N)$ with line segments; this is a well defined subset of R^2. We will refer to the lower boundary of $co(v_1,..., v_N)$ (the subset of the boundary which lies below $co(v_1,..., v_N)$) as the convex envelope of $co(v_1,..., v_N)$. The length of segments of the convex envelope of $co(v_1,..., v_N)$ will refer to the length of the segment when projected onto the *x-axis* or the time length of the segment. It should be clear what is meant by the phrase: Statistics of the convex envelope of the random walk process generated by $v_1,..., v_N$.

We observe that for $\tau = 0$ (3), (4), and (5) give an exact relation between the velocities $v_1,..., v_N$ and the convex envelope of $co(v_1,..., v_N)$. In particular, we can infer the following information: (i) the number of segments in the convex envelope of $co(v_1,..., v_N)$ is exactly the number of aggregates at time $\tau = 0$ given initial velocities $v_1,..., v_N$; (ii) the masses of the aggregates are simply the length of the segments multiplied by the mass of the initial point masses, m; (iii) the ordering of these masses is just the same as the ordering of the segment lengths.

This remarkable correspondence allows one to calculate statistical properties of the mass aggregation at time $\tau = 0$ by calculating statistical properties of the convex envelope of the random walk process generated by the symmetric i.i.d. random variables $v_1,..., v_N$. The following statistical properties of this object are studied in Suidan[(6)] [S]: (i) For $N/2 < n \leqslant N$, the probability of having a segment of length n in the convex envelope is $1/n$, therefore, if $\frac{1}{2} < s < t \leqslant 1$, the probability of finding an aggregate of mass $\mathscr{M}$ with $sM \leqslant \mathscr{M} \leqslant tM$ is $\sum_{sN \leqslant n \leqslant tN} 1/n$ which tends to $\log t/s$ as $N \to \infty$ (a quantity drastically different from the case $\tau < 0$); (ii) given N initial point masses the probability that at $\tau = 0$ we have exactly r aggregates is

$$\frac{1}{r!} \sum_{\substack{n_1 + \cdots + n_r = N \\ n_1,..., n_r > 0}} \frac{1}{n_1 n_2 \cdots n_r} \tag{7}$$

Using the method of generating functions it is easy to show that this expression is equal to

$$\frac{1}{r!\, N!} \frac{d^N}{dz^N} \left(\left(\log \frac{1}{1-z} \right)^r \right) \Bigg|_{z=0} \tag{8}$$

Note that for the statements above we do not need the $4+\delta$ moment condition. Proofs of these statements can be found in [S] which has been submitted to the Annals of Probability. This paper is also available at the Los Alamos National Laboratory Archives at http://www.arXiv.org.

4. FULL MASS AGGREGATE FOR $\tau > 0$

Fix $\tau, \varepsilon > 0$, and $\eta \in (1-\delta/2(4+\delta), 1]$. Given N initial particles, define L_η^N to be the the set of initial velocities $(v_1, ..., v_N) \in R^N$ such that at time τ there is an aggregate of mass Mn/N for some n with the property $\varepsilon N^\eta \leqslant n \leqslant (1-\varepsilon) N$. In this section we prove

Theorem 4. $P(L_\eta^N) \to 0$ as $N \to \infty$.

The probability that there is an aggregate composed of exactly n of the initial particles can be bounded by

$$
\begin{aligned}
&\sum_{j=0}^{N-n} P\left(\prod_{r=0}^{j-1} \Theta(X_{j+1}^n(t) - X_{r+1}^{j-r}(t)) = 1\right) \\
&\qquad \times P\left(\prod_{r=1}^{n-1} \Theta(X_{j+1}^r(t) - X_{j+r+1}^{n-r}(t)) = 1\right) \\
&\qquad \times P\left(\prod_{s=1}^{N-(j+n)} \Theta(X_{j+n+1}^s(t) - X_{j+1}^n(t)) = 1\right) \\
&\qquad \leqslant \sum_{j=0}^{N-n} P\left(\prod_{r=0}^{j-1} \Theta(X_{j+1}^n(t) - X_{r+1}^{j-r}(t)) = 1\right) \\
&\qquad \times P\left(\prod_{s=1}^{N-(j+n)} \Theta(X_{j+n+1}^s(t) - X_{j+1}^n(t)) = 1\right) \\
&\qquad \leqslant \left\{\sum_{j=0}^{(N-n)/2} + \sum_{j=(N-n)/2}^{N-n}\right\} \\
&\qquad \times P\left(\prod_{r=0}^{j-1} \Theta\left(\frac{j-r}{(j-r)+n} \sum_{s=1}^{(j-r)+n} v_{r+s} - \sum_{s=1}^{j-r} v_{r+s} - \frac{mn(j-r)\,\tau}{2}\right) = 1\right) \\
&\qquad \times P\left(\prod_{s=1}^{N-(j+n)} \Theta\left(\frac{n}{n+s} \sum_{i=1}^{n+s} v_{j+i} - \sum_{i=1}^{n} v_{j+i} - \frac{mns\tau}{2}\right) = 1\right)
\end{aligned}
$$

Denote $\sum_1 = \sum_{j=0}^{(N-n)/2}$ and $\sum_2 = \sum_{j=(N-n)/2}^{N-n}$ in the above expressions. We estimate these quantities as follows

$$\sum_1 \leqslant (N-n)\, P\left(\Theta\left(\frac{1}{n+(N-n)/2} \sum_{i=1}^{n+(N-n)/2} v_i - \frac{1}{n} \sum_{i=1}^{n} v_i - \frac{m\tau}{2}\frac{N-n}{2}\right) = 1\right) \tag{9}$$

$$\sum_2 \leqslant (N-n)\, P\left(\Theta\left(\frac{1}{n+(N-n)/2} \sum_{s=1}^{n+(N-n)/2} v_s - \frac{2}{N-n} \sum_{s=1}^{(N-n)/2} v_s - \frac{mn\tau}{2}\right) = 1\right) \tag{10}$$

The bounds for (9) and (10) are similar so we will briefly discuss (9). We bound the first probability in $\sum_1$ by 1; we bound the second probability by recognizing that $N-(j+n) > (N-n)/2$ for the range of j in $\sum_1$ and that $v_1, \ldots, v_N$ are i.i.d. random variables; this allows us to pick the strongest constraint common to all terms in $\sum_1$ and dominate $\sum_1$ by this constraint probability multiplied by $N-n$. The bound in (10) is carried out analogously only switching the probability bounded by 1. Summing over all n with the property that $\varepsilon N^\eta \leqslant n \leqslant (1-\varepsilon) N$, where $\eta \in (1-\delta/2(4+\delta), 1]$, we arrive at the bound

$$\sum_{\varepsilon N^\eta \leqslant n \leqslant (1-\varepsilon) N} \left\{\sum_1 + \sum_2\right\} \leqslant 4N^2 P\left(\frac{1}{N} \sum_{i=1}^{N} v_i \geqslant \frac{M\tau}{4N^{1-\eta}}\right) \tag{11}$$

By similar estimates to those of Section 2 and the fact that we have $4+\delta$ moments the last expression tends to 0 as $N \to \infty$. In light of the results of Section 4, this states that for any fixed $\tau > 0$ the probability of having an aggregate of full mass tends to 1 in the continuum limit. It is important to note that the remaining dust particles must be completely insignificant in the continuum limit. Note that for this argument we do not need the velocity distributions to have densities.

5. UNIFORM CONTROL OF THE LIMITING PROCESS: A SEQUENCE OF TIMES

Fix $\varepsilon, \xi > 0$. We identify a sequence of positive τ_N tending to 0 such that the probability of having an aggregate composed of $n > (1-\varepsilon) N$ of the initial N particles is greater than $1-\xi$; this is the probability of having an aggregate of mass $\mathscr{M} > (1-\varepsilon) M$ by time τ_N be greater than $1-\xi$.

We consider the sequence $\tau_N = \mathscr{D} N^{-(1/2)(\eta-(1-\delta/[2(4+\delta]))}$ when η defined as in the previous section. For an appropriate choice of $\mathscr{D}$ the probability we consider will be bounded from below by $1-\xi$ uniformly in N. To see this we need only replace τ by τ_N in expression (11). We estimate this new quantity using Chebyshev's inequality and the previously mentioned facts regarding moments of the random variables $|\sum_{i=1}^{n} v_i|$ as $n \to \infty$.

$$4N^2 P\left(\frac{1}{N}\sum_{i=1}^{N} v_i \geqslant \frac{M\tau_N}{4N^{1-\eta}}\right) \leqslant \frac{4N^2 \mathscr{R} N^{2+\delta/2}}{(\mathscr{D}N)^{(1/2)(\eta+1)(4+\delta)}}$$

$$= \frac{4\mathscr{R}}{\mathscr{D}^{(1/2)(4+\delta)(\eta+1)} N^{(1/2)(4+\delta)(\eta+1)-(4+\delta/2)}}$$

where $\infty > \mathscr{R} > \limsup E(|\sum_{i=1}^{n} v_i|^{4+\delta})/(n^{2+\delta/2})$. Thus, by making $\mathscr{D}$ large enough we can bound this probability uniformly in N with as much control as necessary; in particular, we can bound it by $\xi/2$ which shows us that our choice of τ_N is satisfactory.

6. BALLISTIC AGGREGATION: $\gamma = 0$

In this section we consider the ballistic case for which $\gamma = 0$; this is the case of no gravity. In this regime, $\tau = -1/\rho t$ for $t > 0$ and N denotes the number of initial particles. Let Ω_N^r be the set of initial velocities $v_1, \ldots, v_N$ such that the final number of aggregates (after the aggregation process has reached completion) is r. Conditions (3), (4), and (5) imply that:

$$P(\Omega_N^r) = \frac{1}{r!} \sum_{\substack{n_1 + \cdots + n_r = N \\ n_1, \ldots, n_r > 0}} \frac{1}{n_1 \cdots n_r}$$

We arrive at this formula by similar methods to those of Section 3; the correspondence with the convex envelope is different only in the sense that the convex envelope gives the long time behavior as opposed to the behavior at time $\tau = 0$. This illustrates that the dynamical properties of the ballistic case are very different from the dynamical properties of the gravitational case. As in Section 3, we remark that the expected number of masses is $\sum_{j=1}^{N} 1/j$; as $N \to \infty$ the number of masses grows as $\log N$.

7. CONCLUDING REMARKS

We have dealt with several questions concerning the sticky particle model with random initial velocities. We showed that in the gravitational case there is a nonrandom time t^* such that after this time the probability of having a mass aggregate of full mass tends to 1 in the continuum limit;

before this time the probability of having an aggregate which has a finite fraction of the total mass of the system tends to 0 in the continuum limit. We referred to the special time t^* as the shock time and have characterized the aggregation process at this shock time. We also found a sequence of times greater than but tending to t^* such that control over the above probabilities is uniform as the number of particles increases. This analysis led to the conclusion that the gravitational case dynamics is dominated by gravity and is essentially nonrandom.

We also dealt with the ballistic case for which $\gamma = 0$. We investigated the long time behavior for a system of N particles with random initial velocities and concluded that this behavior is very different from the behavior in the gravitational case.

APPENDIX: MARCINKIEWICZ–ZYGMUND INEQUALITY AND LARGE DEVIATIONS

In this Appendix we state an inequality due to Marcinkiewicz and Zygmund whose proof uses the remarkable inequality of Khintchine and can be found in Chow and Teicher.[7] We use this inequality to bound the pth moment of the random variable $|\sum_{i=1}^{n} v_i|$ where $v_1,..., v_n,...$ are real valued i.i.d. random variables with $Ev_i = 0$ and finite pth moment; as usual, Ev_i is just the expectation of v_i.

Theorem 5 (Marcinkiewicz–Zygmund). If $\{X_n, n \geqslant 1\}$ are independent random variables with $EX_n = 0$, then for every $p \geqslant 1$ there exist positive constants A_p, B_p depending only upon p for which

$$A_p \left\| \left(\sum_{i=1}^{n} X_i^2 \right)^{1/2} \right\|_p \leqslant \left\| \sum_{i=1}^{n} X_i \right\|_p \leqslant B_p \left\| \left(\sum_{i=1}^{n} X_i^2 \right)^{1/2} \right\|_p \tag{12}$$

The following corollary gives us the desired result.

Corollary 2. If $\{X_n, n \geqslant 1\}$ are i.i.d. random variables with $EX_i = 0$, $E\,|X_1|^p < \infty$, $p \geqslant 2$, and $S_n = \sum_{i=1}^{n} X_i$, then $E\,|S_n|^p = O(n^{p/2})$.

Proof of Corollary 2. If $p > 2$, by Holder's inequality $\sum_{i=1}^{n} X_i^2 \leqslant n^{(p-2)/p} (\sum_{i=1}^{n} |X_i|^p)^{2/p}$ and the conclusion follows from the Marcinkiewicz–Zygmund Inequality.

Using the last fact and the Chebyshev inequality, it is easy to see that the inequalities of Section 2 and 4 are valid; these inequalities are the large deviation type inequalities we refer to in the title of this Appendix.

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